successive differentiation

Statement:- If u and v be two functions of x, possessing derivatives of nth order then

1. $(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_n u v_n$

problem on Leibnitz's theorem

1. If $y = \sin(m\sin^{-1}x)$ prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Soln:-
$$y = \sin(m\sin^{-1}x)$$
or $\sin^{-1}y = m\sin^{-1}x$.

Diff. both side with respect to x we get

$$\frac{1}{\sqrt{1-y^2}}y_1 = m.\frac{1}{\sqrt{1-x^2}}$$

or $y_1\sqrt{1-x^2} = m\sqrt{1-y^2}$

squaring both side we get

$$y_1^2(1-x^2) = m^2(1-y^2)$$

differentiating both side with respect to x we get

$$2y_1y_2(1-x^2) + y_1^2(-2x) = m^2(-2yy_1)$$

(2) or $2y_1y_2(1-x^2) + y_1^2(-2x)$
$= m^2(-2yy_1)$

or $2y_1[y_2(1-x^2) - y_1 x] = 2y_1(-m^2 y)$

or $y_2(1-x^2) - y_1 x = -m^2 y$

or $y_2(1-x^2) - y_1 x + m^2 y = 0$

diff. both side n times with respect to x by Leibnitz's theorem we get

$[y_2(1-x^2)]_n - [y_1 x]_n + m^2[y]_n = 0$

or $[y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2)$ + all other terms vanish$] - [y_{n+1}x + {}^nC_1 y_n \cdot 1 +$ all other terms vanish$] + m^2 y_n = 0$

or $y_{n+2}(1-x^2) + n y_{n+1}(-2x) + \frac{n(n-1)}{1\cdot 2} y_n(-2) - y_{n+1}x - n y_n + m^2 y_n = 0$

or $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} + (-n^2 + n - n + m^2)y_n = 0$

Hence $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} + (m^2 - n^2)y_n = 0$

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part I

partial differentiation

(3) Euler's theorem

Statement:- If $u = f(x,y)$ be a homogeneous function of two independent variable x and y and of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

problem (1) If $u = \sec^{-1}\frac{x^2+y^2}{x-y}$

prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cot u$$

Solution:- $u = \sec^{-1}\frac{x^2+y^2}{x-y}$

or $\sec u = \frac{x^2+y^2}{x-y}$

or $\sec u = \frac{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}{x\left(1-\frac{y}{x}\right)}$

or $\sec u = \frac{x\left[1+\left(\frac{y}{x}\right)^2\right]}{1-\frac{y}{x}} = v$ (say)

we find that v is a homogeneous function of two independent variable x and y and of degree 1. Hence applying Euler's theorem for function v we get

(4) $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 1.v = v$ — (2)

From (1) $v = \sec u$

$\therefore \frac{\partial v}{\partial x} = \sec u \tan u \frac{\partial u}{\partial x}$

and $\frac{\partial v}{\partial y} = \sec u \tan u \frac{\partial u}{\partial y}$

Substituting these values in (2) we get

$x \sec u \tan u \frac{\partial u}{\partial x} + y \sec u \tan u \frac{\partial u}{\partial y} = \sec u$

or $\sec u \tan u \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) = \sec u$

~~or~~ or $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\cot} u$

(5) <u>problem</u> (2) if $v = \tan^{-1}\dfrac{x^3+y^3}{x-y}$ then $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = \sin 2v$

Soln:- we have $v = \tan^{-1}\dfrac{x^3+y^3}{x-y}$

or $\tan v = \dfrac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left(1-\frac{y}{x}\right)}$

& $\tan v = \dfrac{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}{1-\frac{y}{x}} = u$ (say) — (1)

we find that u is a homogeneous function of two independent variable x and y and of degree two. Hence applying Euler's theorem for the function u we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 2u \quad \text{— (2)}$$

From (1) $u = \tan v$

$$\therefore \frac{\partial u}{\partial x} = \sec^2 v\,\frac{\partial v}{\partial x}$$

and $$\frac{\partial u}{\partial y} = \sec^2 v\,\frac{\partial v}{\partial y}$$

Substituting these two values in (2) we get

(6) $x\sec^2 v \frac{\partial v}{\partial x} + y \sec^2 v \frac{\partial v}{\partial y} = 2\tan v$

or $\sec^2 v \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 2\frac{\sin v}{\cos v}$

or $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2\frac{\sin v}{\cos v}\cos^2 v$

$= \sin 2v$

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